

Advanced Engineering Mathematics
Lecture Note 11

1 Integral Calculus

Comparison test type 1. If f and g be two positive functions on $[a, b]$ and $f(x) \leq g(x) \forall x \in [a, b]$, then

- i) $\int_a^b f(x)$ converges if $\int_a^b g(x)$ converges.
- ii) $\int_a^b g(x)$ diverges if $\int_a^b f(x)$ diverges.

Comparison test type 2. If f and g be two positive functions on $[a, \infty)$ and $f(x) \leq g(x) \forall x \in [a, \infty)$, then

- i) $\int_a^\infty f(x)$ converges if $\int_a^\infty g(x)$ converges.
- ii) $\int_a^\infty g(x)$ diverges if $\int_a^\infty f(x)$ diverges.

Limit form test. If f and g be two positive functions on $[a, b]$ and $f(x) \leq g(x) \forall x \in [a, b]$ and $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = l (\neq 0, < \infty)$, then $\int_a^b f(x)$ and $\int_a^b g(x)$ converge or diverge together at a .

Example 1.1. $I = \int_0^1 \frac{e^{-x}}{x^{1-n}} dx$

Sol. Here 0 is the only point of infinite discontinuity of the integrand if $n < 1$. Let $f(x) = \frac{e^{-x}}{x^{1-n}}$ and $g(x) = \frac{1}{x^{1-n}}$, then

$$\lim_{x \rightarrow 0^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0^+} \frac{\frac{e^{-x}}{x^{1-n}}}{\frac{1}{x^{1-n}}} = \lim_{x \rightarrow 0^+} e^{-x} = 1.$$

Now, $\int_0^1 \frac{dx}{x^{1-n}}$ converges if $1-n < 1 \Rightarrow n > 0$. Hence the given integral I converges if $n > 0$.

Example 1.2. $I = \int_0^\infty \frac{\cos x}{1+x^2} dx$

Sol. The right end point b is infinite. Choose $f(x) = \frac{\cos x}{1+x^2}$ and $g(x) = \frac{1}{1+x^2}$. Then $f(x) = \frac{\cos x}{1+x^2} \leq \frac{1}{1+x^2} = g(x) \forall x \in [0, \infty)$. Now, $I_1 = \int_0^\infty g(x) dx = \int_0^\infty \frac{dx}{1+x^2}$ is convergent. Therefore, by comparison test type 1, $I = \int_0^\infty f(x) dx$ is convergent.

Beta function. $B(m, n) = \beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$. Hence, $B(m, n)$ is proper integral if $m \geq 1, n \geq 1$. Improperness is attained when $m, n < 1$.

$$\int_0^1 x^{m-1}(1-x)^{n-1} dx, \quad m < 1, n < 1$$

we will obtain $B(m, n)$ converges if and only if $m, n > 0$.

Gamma function. $\Gamma = \int_0^\infty e^{-x} x^{n-1} dx$ converges if and only if $n > 0$. Here, 0 is a point of infinite discontinuity if $n < 1$.

$$\Gamma = \int_0^\infty e^{-x} x^{n-1} dx = \int_0^1 e^{-x} x^{n-1} dx + \int_1^\infty e^{-x} x^{n-1} dx$$

we can show that $\Gamma = \int_0^\infty e^{-x} x^{n-1} dx$ is convergent if only if $n > 0$.

Properties of Beta and Gamma functions.

1. $\Gamma(n) = (n-1)\Gamma(n-1)$ when $n > 1$

$$\begin{aligned}\Gamma(n) &= \int_0^\infty e^{-x} x^{n-1} dx = [-x^{n-1} e^{-x}]_0^\infty + (n-1) \int_0^\infty e^{-x} x^{n-2} dx \\ &= (n-1)\Gamma(n-1) \\ \Rightarrow \Gamma(n) &= (n-1)\Gamma(n-1) \\ &= (n-1)(n-2) \cdots 2.1 \\ &= (n-1)!\end{aligned}$$

2. $B(m, n) = B(n, m)$

$$\begin{aligned}B(m, n) &= \int_0^1 x^{m-1} (1-x)^{n-1} dx \\ &= \int_0^1 (1-y)^{m-1} y^{n-1} dy \quad [Let, (1-x) = y] \\ &= B(n, m)\end{aligned}$$

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3. $B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, m > 0, n > 0.$

4. $\sqrt{\pi}\Gamma(2m) = 2^{2m-1}\Gamma(m)\Gamma(m + \frac{1}{2}), m > 0 \Rightarrow$ Legendre's Duplication Formula.

5. $B(m, n) = 2^{1-2m}B(m, \frac{1}{2})$

Example 1.1. Find the value of $\Gamma(\frac{1}{2})$.

Soln. Let $m = n = \frac{1}{2}$, then using properties of Beta and Gamma functions, we get

$$B(\frac{1}{2}, \frac{1}{2}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} + \frac{1}{2})} \Rightarrow [\Gamma(\frac{1}{2})]^2 = B(\frac{1}{2}, \frac{1}{2})$$

Again,

$$\begin{aligned} B(\frac{1}{2}, \frac{1}{2}) &= \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}dx = \int_0^1 \frac{dx}{\sqrt{x}\sqrt{1-x}} \quad [x = \sin^2(\theta) \Rightarrow dx = 2 \sin \theta \cos \theta d\theta] \\ &= 2[\theta]_0^{\frac{\pi}{2}} \\ &= \pi \end{aligned}$$

Hence, $\Gamma(\frac{1}{2}) = \sqrt{B(1, 1)} = \sqrt{\pi}$

Example 1.2. Evaluate $\int_0^\infty e^{-x^2} dx$.

Soln. Let $x^2 = z \Rightarrow 2x dx = dz$. So we get,

$$\begin{aligned} I &= \int_0^\infty e^{-x^2} dx \\ &= \frac{1}{2} \int_0^\infty e^{-z} z^{-\frac{1}{2}} dz \\ &= \frac{1}{2} \int_0^\infty e^{-z} z^{\frac{1}{2}-1} dz \\ &= \frac{1}{2} \Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2} \end{aligned}$$

Example 1.3. Evaluate $I = \int_0^1 \frac{xdx}{\sqrt{1-x^5}}$

Soln. Let us take,

$$\begin{aligned} x^5 &= z \Rightarrow 5x^4 dx = dz \\ &\Rightarrow dx = \frac{dz}{5x^4} \\ &\Rightarrow dx = \frac{1}{5} z^{-\frac{4}{5}} dz \end{aligned}$$

Now,

$$\begin{aligned}
 I &= \int_0^1 z^{\frac{1}{5}} \frac{1}{5} z^{-\frac{4}{5}} (1-z)^{-\frac{1}{2}} dz \\
 &= \frac{1}{5} \int_0^1 z^{-\frac{3}{5}} (1-z)^{-\frac{1}{2}} dz \\
 &= \frac{1}{5} \int_0^1 z^{\frac{2}{5}-1} (1-z)^{\frac{1}{2}-1} dz \\
 &= B\left(\frac{2}{5}, \frac{1}{2}\right)
 \end{aligned}$$

Example 1.4. Evaluate $I = \int_0^1 \frac{dx}{\sqrt{1-x^4}}$ in terms of Gamma function.

Soln. Let us take,

$$\begin{aligned}
 x^4 = z &\Rightarrow 4x^3 dx = dz \\
 &\Rightarrow dx = \frac{dz}{4x^3} \\
 &\Rightarrow dx = \frac{1}{4} z^{-\frac{3}{4}} dz
 \end{aligned}$$

Now,

$$\begin{aligned}
 I &= \int_0^1 \frac{1}{4} z^{-\frac{3}{4}} dz \frac{1}{\sqrt{1-z}} \\
 &= \frac{1}{4} \int_0^1 z^{\frac{1}{4}-1} (1-z)^{\frac{1}{2}-1} dz \\
 &= \frac{1}{4} B\left(\frac{1}{4}, \frac{1}{2}\right) \\
 &= \frac{1}{4} \frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4}+\frac{1}{2})} = \frac{1}{4} \frac{\sqrt{\pi}\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} = \frac{\sqrt{\pi}}{4} \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} = \frac{1}{8} \sqrt{\frac{2}{\pi}} [\Gamma(\frac{1}{4})]^2
 \end{aligned}$$

$$[\Gamma(\frac{3}{4})\Gamma(\frac{1}{4}) = \sqrt{2\pi}]$$

Differentiation under integral sign. Suppose f is defined on a rectangle $R : [a, b] \times [c, d]$ and it is continuous on R . Let $f_x(x, y)$ exists and continuous on R , then the integral

$$g(x) = \int_c^d f(x, y) dy$$

is differentiable with respect to x on (a, b) and

$$g'(x) = \int_c^d f_x(x, y) dy$$

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Leibnitz Rule of Differentiation. Let $f(x, y)$ and $f_x(x, y)$ are continuous on a rectangle $R = [a, b] \times [c, d]$. Let $\phi, \varphi : [a, b] \Rightarrow [c, d]$ be both differentiable on (a, b) . Then the integral

$$g(x) = \int_{\phi(x)}^{\varphi(x)} f(x, y) dy$$

is differentiable on (a, b) and

$$g'(x) = \int_{\phi(x)}^{\varphi(x)} f_x(x, y) dy + f(x, \varphi(x))\varphi'(x) - f(x, \phi(x))\phi'(x)$$

Example 1.1. Evaluate by diff. under integral sign, the following integral $I = \int_0^1 \frac{x^y - 1}{\log x} dx$.

Soln. Let,

$$\begin{aligned}\phi(y) &= \int_0^1 \frac{x^y - 1}{\log x} dx. \quad (\text{Using Leibnitz rule of derivative}) \\ \Rightarrow \phi'(y) &= \int_0^1 \frac{\partial}{\partial y} \left(\frac{x^y - 1}{\log x} \right) dx = \int_0^1 \frac{1}{\log x} \frac{\partial}{\partial y} (x^y - 1) dx = \int_0^1 \frac{1}{\log x} x^y \log x dx \\ \Rightarrow \phi'(y) &= \int_0^1 x^y dx = \left[\frac{x^{y+1}}{y+1} \right]_0^1 = \frac{1}{y+1} \\ \Rightarrow \phi(y) &= \int \frac{1}{y+1} dy = \log_e y + c\end{aligned}$$

Again we know that,

$$\begin{aligned}\phi(y) &= \int_0^1 \frac{x^y - 1}{\log x} dx \Rightarrow \phi(0) = \int_0^1 0 dx = 0 \\ \phi(0) &= \log_e 0 + 1 + c \Rightarrow 0 = 0 + c \Rightarrow c = 0\end{aligned}$$

The required value of the integral is $\phi(y) = \log(1 + y)$.

Example 1.2. Evaluate $I = \int_0^\infty e^{-x^2} \cos \alpha x dx$.

Soln. Let,

$$\begin{aligned}I(\alpha) &= \int_0^\infty e^{-x^2} \cos \alpha x dx \\ \Rightarrow \frac{dI}{d\alpha} &= - \int_0^\infty x e^{-x^2} \sin \alpha x dx \\ &= \left[\frac{1}{2} e^{-x^2} \sin \alpha x \right]_0^\infty - \frac{\alpha}{2} \int_0^\infty e^{-x^2} \cos \alpha x dx \quad (\text{Integrating by parts}) \\ &= \frac{\alpha}{2} I(\alpha) \\ \Rightarrow \frac{d\alpha}{I} &= -\frac{\alpha}{2} d\alpha \Rightarrow \log I(\alpha) = -\frac{\alpha^2}{4} + c \\ &\Rightarrow I(\alpha) = c e^{-\frac{\alpha^2}{4}}\end{aligned} \tag{1.1}$$

Since, $I(\alpha) = \int_0^\infty e^{-x^2} \cos \alpha x \, dx \Rightarrow I(0) = \int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}$.

Now, from (1.1) $\frac{\sqrt{\pi}}{2} = I(0) = ce^0 = c \Rightarrow c = \frac{\sqrt{\pi}}{2}$. Therefore, the required value of the integral is $I(\alpha) = \frac{\sqrt{\pi}}{2} e^{-\frac{\pi^2}{4}}$.

Integration in $\mathbb{R}^2$ (double integral) and in $\mathbb{R}^3$ (triple integral).

$$f : [a, b] \Rightarrow \mathbb{R}, \quad I = \int_a^b f(x) \, dx$$

$$f : [a, b] \times [c, d] \Rightarrow \mathbb{R}, \quad I_1 = \int_a^b \int_c^d f(x, y) \, dx \, dy$$

$$f : [a, b] \times [c, d] \times [e, f] \Rightarrow \mathbb{R}, \quad I_1 = \int_a^b \int_c^d \int_e^f f(x, y, z) \, dx \, dy \, dz$$

Example 1.3. Evaluate $I = \int_c \frac{dx}{x+y}$, where c is the curve $x = at^2, y = 2at, 0 \leq t \leq 2$.

Soln. We have, $x = at^2, y = 2at \Rightarrow dx = 2at \, dt, dy = 2a \, dt$. Now,

$$\begin{aligned} I &= \int_0^2 \frac{2at \, dt}{at^2 + 2at} \\ &= 2 \int_0^2 \frac{dt}{t+2} \\ &= 2 [\log(t+2)]_0^2 = 2 [\log 4 - \log 2] = 2 \log 2 \end{aligned}$$

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Line integral. A curve in $\mathbb{R}^2$ is given by $c = \{(x, y) : x = \phi(t), y = \psi(t), a \leq t \leq b\}$. We may write this curve C as $x = \phi(t), y = \psi(t), a \leq t \leq b$. Let $f(x, y)$ be a bounded function defined at every point of C . Consider any partition P of $[a, b]$ such that $P = \{a = t_0, t_1, t_2, \dots, t_n = b\}$. Let $x_r = \phi(t_r), y_r = \psi(t_r)$ and $\xi_r \in [t_{r-1}, t_r]$ be arbitrary. consider the sum

$$S = \sum_{r=1}^n f[\phi(\xi_r), \psi(\xi_r)](x_r - x_{r-1}).$$

If the norm $\mu(P) \rightarrow 0$, the sum S tends to a finite limit which is independent of the choice of the point ξ_r and the limit is denoted by $\int_C f(x, y) dx$.

Working principle :
$$\int_C f(x, y) dx = \int_a^b f(\phi(t), \psi(t))\phi'(t) dt$$

Example 1.1. Find $I = \int_C \frac{y^2 dx - x^2 dy}{x^2 + y^2}$, where C is the semi-circle $x = a \cos t, y = a \sin t, 0 \leq t \leq \pi$.

Sol.

$$\begin{aligned} I &= \int_C \frac{y^2 dx - x^2 dy}{x^2 + y^2} = \int_0^\pi \frac{y^2 \frac{dx}{dt} - x^2 \frac{dy}{dt}}{x^2 + y^2} dt \\ &= \frac{1}{a^2} \int_{t=0}^\pi [a^2 \sin^2 t (-a \sin t) - a^2 \cos^2 t (a \cos t)] dt \\ &= -\frac{a^3}{a^2} \int_0^\pi [\sin^3 t + \cos^3 t] dt \\ &= -a \int_0^\pi [\sin^3 t + \cos^3 t] dt = -\frac{4a}{5} \end{aligned}$$

Example 1.2. Evaluate $I = \int_C (x^2 dx + xy dy)$ taken along the line segment $(1, 0)$ to $(0, 1)$.

Sol. The equation of the line passing through $(1, 0)$ to $(0, 1)$ is $x + y = 1 \Rightarrow dy = -dx$.

Then,
$$I = \int_C (x^2 dx + xy dy) = \int_0^1 (x^2 dx + x(1-x)(-dx)) = \int_0^1 (2x^2 - x) dx = -\frac{1}{6}.$$

Double integral over a rectangle R . Let f be a function of two variables x and y over a rectangle $R : [a, b] \times [c, d]$. Let

$$P_1 = \{a = x_0, x_1, \dots, x_n = b\}, P_2 = \{c = y_0, y_1, \dots, y_n = d\}$$

$$U(P, f) = \sum_i \sum_j M_{ij}(x_i - x_{i-1})(y_j - y_{j-1}), L(P, f) = \sum_i \sum_j m_{ij}(x_i - x_{i-1})(y_j - y_{j-1}),$$

where $U(P, f)$ and $L(P, f)$ are the upper sum and lower sum, respectively. M_{ij} and m_{ij} are the upper bound and the lower bound of $f(x, y)$ in $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$, respectively.

If $\lim_{n \rightarrow \infty} U(P, f)$ exists, then it is denoted as $\iint_R^+ f(x, y) \, dx \, dy$ (upper integral) and similarly if $\lim_{n \rightarrow \infty} L(P, f)$ exists, then it is denoted as $\iint_R^- f(x, y) \, dx \, dy$ (lower integral).

If both upper integral and lower integral exists and equal, then the value is denoted by $\iint_R f(x, y) \, dx \, dy$. This is called the double integral of f over R .

Example 1.3. Evaluate $\iint_R (x^2 + 2y) \, dx \, dy$ where $R = [0, 1] \times [0, 2]$.

Sol.

$$\int_{y=0}^2 \left[\int_{x=0}^1 (x + 2x) \, dx \right] dy = \int_0^2 \left[\frac{x^3}{3} + 2xy \right]_0^1 dy = \int_0^2 \left[\frac{1}{3} + 2y \right]_0^1 dy = \left[\frac{y}{3} + y^2 \right]_0^2 = \frac{14}{3}.$$

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Example 1.1. Evaluate $I = \int_R x \sin(x+y) dx dy$, where $R := [0, \pi] \times [0, \frac{\pi}{2}]$.

Sol. Using $\left[\int_{x=a}^b \left\{ \int_{y=c}^d f(x,y) dy \right\} dx = \int_{y=c}^d \left\{ \int_{x=a}^b f(x,y) dx \right\} dy \right]$, we have

$$\begin{aligned} I &= \int_{y=0}^{\frac{\pi}{2}} \int_{x=0}^{\pi} x \sin(x+y) dx dy \\ &= \int_{x=0}^{\pi} \left[\int_{y=0}^{\frac{\pi}{2}} x \sin(x+y) dy \right] dx \\ &= \int_0^{\pi} \left[-x \cos(x+y) \right]_0^{\frac{\pi}{2}} dx \\ &= \int_0^{\pi} [x \sin x + x \cos x] dx \\ &= \pi - 2 \end{aligned}$$

Double integral over a region. If f is a continuous function in a domain D which is bounded by the curve $y = \phi(x), y = \psi(x), x = a, y = b$, where ϕ and ψ are continuous on $[a, b]$ and $\phi(x) \leq \psi(x)$ then,

$$\iint_D f(x,y) dx dy = \int_{x=a}^b \left[\int_{y=\phi(x)}^{\psi(x)} f(x,y) dy \right] dx.$$

Example 1.2. Evaluate $\iint_R x^3 y^2 dx dy$ over the circle $C : x^2 + y^2 \leq a^2$.

Sol. The circle C is

$$x^2 + y^2 \leq a^2 \Rightarrow y^2 \leq a^2 - x^2 \Rightarrow -\sqrt{a^2 - x^2} \leq y \leq \sqrt{a^2 - x^2}.$$

When $y = 0$, then $x^2 \leq a^2 \Rightarrow -a \leq x \leq a$.

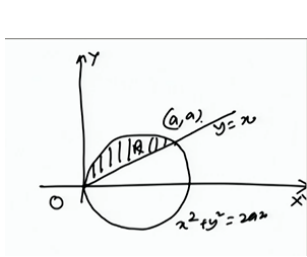
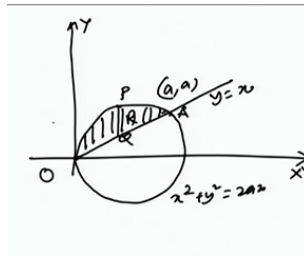
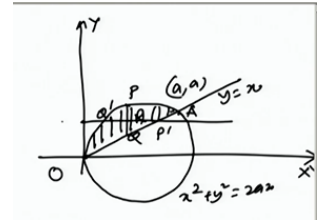
$$\begin{aligned} \iint_R x^3 y^2 dx dy &= \int_{x=-a}^a x^3 \left\{ \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2 dy \right\} dx \\ &= \frac{2}{3} \int_{-a}^a x^3 (a^2 - x^2)^{\frac{3}{2}} dx \\ &= 0 \end{aligned}$$

Example 1.3. Evaluate $I = \iint_R y dx dy$ over the part of the plane bounded by the line $y = x$ and the parabola $y = 4x - x^2$.

Sol. Putting $y = x$ in $y = 4x - x^2 \Rightarrow x = 4x - x^2 \Rightarrow x(x-3) = 0 \Rightarrow x = 0, 3$. The line joining the parabola meets at $(0, 0)$ and $(3, 3)$. Any line parallel to y -axis cuts the boundary of the region into two points, say P and Q . Thus,

$$I = \iint_R y dx dy = \int_{x=0}^3 dx \int_x^{4x-x^2} y dy = \frac{1}{2} \int_0^3 [(4x-x^2)^2 - x^2] dx = \frac{54}{5}.$$

Example 1.4. Evaluate $I = \iint_R \cos(x+y) dx dy$ over the domain closed by $x = 0, y = \pi, y = x$.



Sol.

$$\begin{aligned}
 I &= \int_{x=0}^{\pi} \int_{y=x}^{\pi} \cos(x+y) dx dy \\
 &= \int_{x=0}^{\pi} dx \left\{ \int_{y=x}^{\pi} \cos(x+y) dy \right\} \\
 &= \int_{x=0}^{\pi} \left[\sin(x+y) \right]_{y=x}^{\pi} dx \\
 &= \int_{x=0}^{\pi} (-\sin x - \sin 2x) dx \\
 &= -2
 \end{aligned}$$

Change of the order of the integration.

Example 1.5. Change the order of integration in the double integral $I = \int_0^a \left\{ \int_x^{\sqrt{2ax-x^2}} f(x, y) dy \right\} dx$.

Sol. The given domain of the integration is described by a line which starts from $x = 0$ i.e., y -axis and moving parallel to y -axis terminates at $x = a$. Further the extremities of the moving line lie on the parts of the line $y = x$ and the circle $x^2 + y^2 = 2ax$ in the first quadrant. When we change the order of integration, the same region is described by a line moving parallel to x -axis instead of y -axis. The line $y = x$ and the circle $x^2 + y^2 = 2ax$ cut at the points $(0, 0)$ and (a, a) . Any line parallel to x -axis cuts the domain into two points $P'Q'$. We have

$$x^2 - 2ax + y^2 = 0 \Rightarrow x = a \pm \sqrt{a^2 - y^2}.$$

Then,

$$I = \int_{y=0}^a dy \int_{x=a-\sqrt{a^2-y^2}}^y f(x, y) dx.$$

Example 1.6. $\int_0^1 dx \int_x^{\sqrt{x}} f(x, y) dy \rightarrow \int_0^1 dy \int_{y^2}^y f(x, y) dx$

Example 1.7. $\int_{-1}^1 dx \int_0^{\sqrt{1-x^2}} f(x, y) dy \rightarrow \int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y) dx.$